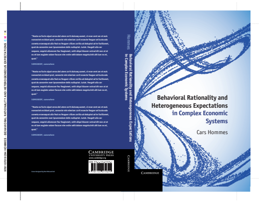
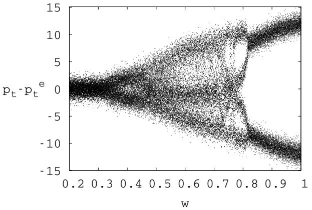
Complex Systems Workshop
Lecture II: Non-linear Cobweb Model

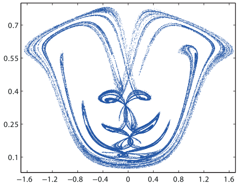
Cars Hommes

CeNDEF, University of Amsterdam

CEF 2013, July 9, Vancouver







Outline

1. Nonlinear Cobweb model with Homogeneous Beliefs (Chapter 4)
   - Naive expectations
   - Rational expectations
   - Adaptive expectations
2. Cobweb Model with Heterogeneous Beliefs (Chapter 5)
   - Rational versus naive
   - Evolutinoary Selection and Reinforcement Learning
   - Fundamentalists versus naive
   - Contrarians versus naive
   - SAC learning versus naive

Literature

- Hommes, C.H., (2013), Behavioral Rationality and Heterogeneous Expectations in Complex Economic Systems, Cambridge.
- Hommes, C.H., (1994), Dynamics of the cobweb model with adaptive expectations and nonlinear supply and demand, *Journal of Economic Behavior and Organization* 24, 315-335.
- Brock, W.A. and Hommes, C.H. (1997), A rational route to randomness, *Econometrica*, 65, 1059-1095.

Cobweb ('hog cycle') Model

- market for non-storable consumption good (e.g. corn, hogs)
- **production lag**; producers form price expectations one period ahead
- partial equilibrium; market clearing prices

p_t^e: producers' price expectation for period t
p_t: realized market equilibrium price p_t

Cobweb ('hog cycle') Model (continued)

$$D(p_t) = a - dp_t(+\epsilon_t) \qquad a \in R, \quad d \geq 0 \qquad \text{demand} \qquad (1)$$

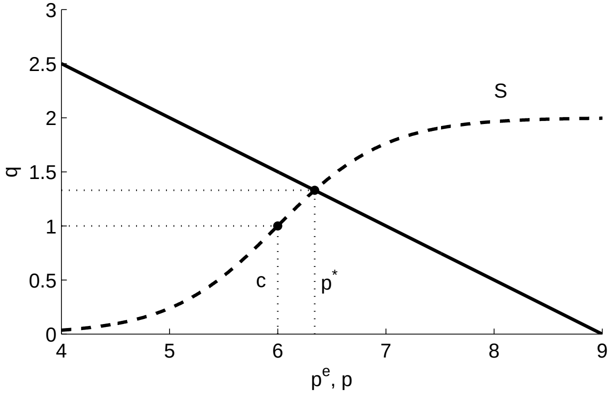
$$S_\lambda(p_t^e) = \tanh(\lambda(p_t^e - 6)) + 1, \qquad \lambda > 0, \qquad \text{supply} \qquad (2)$$

$$D(p_t) = S_\lambda(p_t^e) \qquad \text{market clearing} \qquad (3)$$

$$p_t^e = H(p_{t-1}, ..., p_{t-L}), \qquad \text{expectations} \qquad (4)$$

Price dynamics: $p_t = D^{-1} S_\lambda(H(p_{t-1}, ..., p_{t-L}))$
Expectations Feedback System:
dynamical behavior depends upon expectations hypothesis;
supply driven, **negative feedback**

Demand and (nonlinear) Supply in Cobweb Model

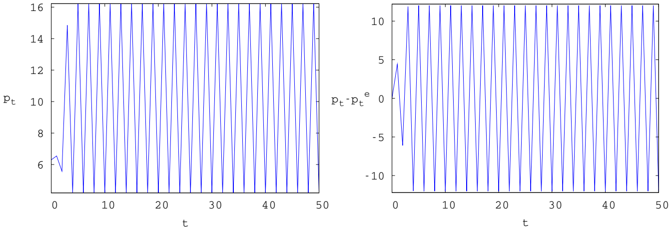


Expectations

- naive expectations: $p_t^e = p_{t-1}$
- adaptive expectations; $p_t^e = wp_{t-1} + (1-w)p_{t-1}^e$
- backward looking average expectations $p_t^e = w_1 p_{t-1} + wp_{t-2}$
- rational expectations: $p_t^e = E_t[p_t] = p^*$

Naive Expectations Benchmark ($p_t^e = p_{t-1}$)

unstable steady state iff $S'(p^*)/D'(p*) < -1$



Regular **period 2 price cycle** with **systematic forecasting errors**

Agents will **learn** from their mistakes and **adapt forecasting behavior**

Rational Expectations (Muth, 1961)

Expectations are **model consistent**
all agents are rational and **compute** expectations from market equilibrium equations

$$p_t^e = E_t[p_t] \qquad \text{or} \qquad p_t^e = p_t \qquad \text{or} \qquad p_t^e = p^*$$

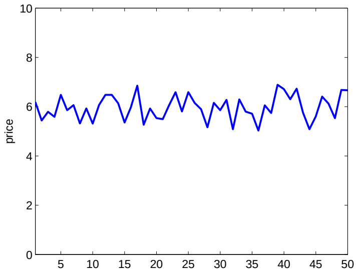
implied **self-fulfilling** RE price dynamics

$$p_t = p^* + \delta_t$$

perfect foresight, **no systematic forecasting errors**

Important Note: this is impossible in complex, heterogeneous world

Rational Expectations Benchmark ($p^* = 5.93$)



Problem: need perfect knowledge of "law of motion" and high computing abilities

Adaptive Expectations (“error learning”)

Nerlove 1958

$$
\begin{aligned}
p_t^e &= (1-w)p_{t-1}^e + w p_{t-1} \\
&= p_{t-1}^e + w(p_{t-1} - p_{t-1}^e) \\
&= w p_{t-1} + (1-w) w p_{t-2} + \cdots (1-w)^{j-1} w p_{t-j} + \cdots
\end{aligned}
$$

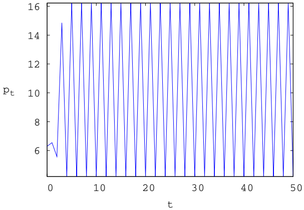
weighted average of past prices

1-D (expected) price dynamics: $p_t^e = wD^{-1}S(p_{t-1}^e) + (1-w)p_{t-1}^e$

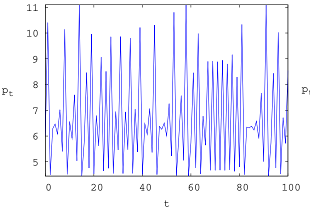
stable steady state if $-\frac{2}{w} + 1 < \frac{S'(p^*)}{D'(p^*)} (< 0)$

- more **stabilizing** in **linear** models, but
- possibly low amplitude **chaos** in nonlinear models

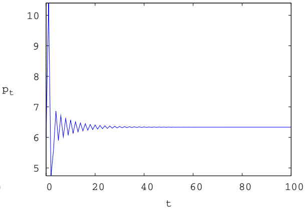
Adaptive Expectations may lead to Chaos



$w = 1$



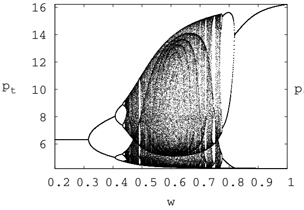
$w = 0.5$



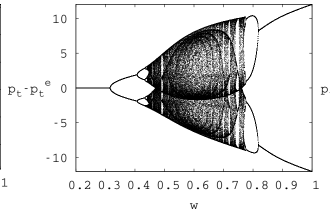
$w = 0.3$

weighted average of nonlinear monotonic D/S curves leads to non-monotonic chaotic map

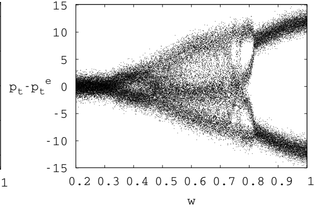
Adaptive Expectations lead to Chaotic Forecast Errors



chaotic prices



chaotic errors



noisy errors

In a **nonlinear** world, adaptive expectations may lead to (small) **chaotic** forecasting errors

Non-monotonic chaotic map for monotonic D and S

C.H. Hommes / Journal of Economic Behavior and Organization 24 (1994) 315–335 327

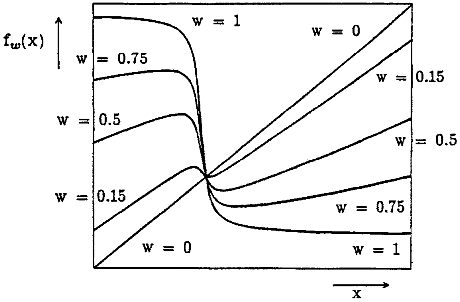


Fig. 4. Graphs of the map f_w for different values of the expectations weight factor w, with $a=0.8$, $b=0.25$, and $\lambda=4$. For w close to 0 f_w has a globally stable equilibrium. For w close to 1 f_w has a stable period 2 cycle. For w close to 0.5 the map f_w is chaotic.

the parameter a, infinitely many period doubling bifurcations occur as w is

Adaptive Expectations in a Nonlinear World

- adaptive expectations is **stabilizing** in the sense that it reduces the amplitude of price fluctuations and forecast errors
- small amplitude **chaotic** price fluctuations may arise around the unstable steady state
- forecast **errors** may be **chaotic**, highly irregular, with little systematic structure
- in a **nonlinear** world, adaptive expectations may be a **behaviorally rational** strategy for **boundedly rational** agents

Ken Arrow on Heterogeneous Expectations

"*One of the things that microeconomics teaches you is that individuals are not alike. There is* **heterogeneity**, *and probably* **the most important heterogeneity here is heterogeneity of expectations**. *If we didn't have heterogeneity, there would be no trade. But developing an analytic model with heterogeneous agents is difficult.*"

(Ken Arrow, In: D. Colander, R.P.F. Holt and J. Barkley Rosser (eds.), The Changing Face of Economics. Conversations with Cutting Edge Economists. The University of Michigan Press, Ann Arbor, 2004, p. 301.)

Cobweb Model with Homogeneous Expectations

- Demand: $D(p_t) = a - dp_t(+\epsilon_t)$, $a \in R$, $d \geq 0$.
- Supply: $S_\lambda(p_t^e) = sp_t^e$, $s > 0$.
- Market clearing: $D(p_t) = S_\lambda(p_t^e)$.
- Expectations: $p_t^e = H(p_{t-1}, ..., p_{t-L})$.

- **Price dynamics**: $p_t = D^{-1}S_\lambda(H(p_{t-1}, ..., p_{t-L}))$.
- Note: linear supply curve derived from profit maximization with **quadratic cost function** $c(q) = q^2/(2s)$.

Cobweb Model with Heterogeneous Beliefs I.

- Market clearing:

$$a - dp_t = n_{1t} s p^e_{1t} + n_{2t} s p^e_{2t} \quad (+\epsilon_t),$$

  where n_{1t} and $n_{2t} = 1 - n_{1t}$ are **fractions** of the two types.

- Forecasting rules:
  1. **rational**: $p^e_{1t} = p_t$,
  2. **naive**: $p^e_{2t} = p_{t-1}$.

- Information gathering **costs**: rational - $C > 0$; naive - free.

Cobweb Model with Heterogeneous Beliefs II.

- **Market clearing** becomes:

$$a - dp_t = n_{1t} s p_t + n_{2t} s p_{t-1} \qquad (+\epsilon_t)$$

- **Price dynamics**:

$$p_t = \frac{a - n_{2t} s p_{t-1}}{d + n_{1t} s}.$$

- How do **fractions** change over time?

Evolutionary or Reinforcement Learning

- Agents can choose between different types of forecasting rules.
- Sophisticated rules may come at information gathering costs $C > 0$ (Simon, 1957), simple rules are freely available.
- Agents evaluate the net past performance of all rules, and **tend to follow rules that have performed better in the recent past**.
- Evolutionary fitness measure $\equiv$ past realized net profits.

Discrete Choice Model

- **Fitness Measure**: random utility

$$\tilde{U}_{ht} = U_{ht} + \epsilon_{iht},$$

U_{ht}: deterministic part of fitness measure,
ϵ_{iht}: idiosyncratic noise, IID, extreme value distr.

Fractions of Belief Types

- **Discrete choice** or **multi-nomial logit** model:

$$n_{ht} = e^{\beta U_{h,t-1}} \ / \ Z_{t-1},$$

  where $Z_{t-1} = \sum e^{\beta U_{h,t-1}}$ is a normalization factor.

- β is the **intensity of choice**, inversely related to SD idiosyncratic noise: $\beta \sim 1/\sigma$.

- $\beta = 0$: all types equal weight (random choice).
- $\beta = \infty$: "neoclassical limit", i.e. **all** agents choose **best** predictor.

Fitness Measure

- **Evolutionary Fitness Measure**:
  weighted average of **past realized net profits**

$$U_{ht} = \pi_{ht} + wU_{h,t-1}$$

- π_{ht} net realized profit (minus costs) strategy h.
- w measures **memory strength**
  - $w = 1$: infinite memory; fitness $\equiv$ accumulated profits,
  - $w = 0$: memory one lag; fitness most recently realized net profit.

Fitness Measure Profits

- **Profits** of type h:

$$\pi_{ht} = p_t s p^e_{ht} - \frac{(s p^e_{ht})^2}{2s}.$$

- Profits of **rational** agents:

$$\pi_{1t} = \frac{s}{2} p_t^2 - C$$

- Profits of **naive** agents:

$$\pi_{2t} = \frac{s}{2} p_{t-1}(2p_t - p_{t-1})$$

Profit difference

- **Difference** in profits:

$$\pi_{1t} - \pi_{2t} = \frac{s}{2}(p_t - p_{t-1})^2 - C.$$

- **difference** in fractions:

$$m_{t+1} = n_{1,t+1} - n_{2,t+1} \quad = \mathrm{Tanh}(\tfrac{\beta}{2}[\pi_{1t} - \pi_{2t}])$$

$$= \mathrm{Tanh}(\tfrac{\beta}{2}[\tfrac{s}{2}(p_t - p_{t-1})^2 - C])$$

- When the **costs** for rational expectations **outweigh** the **forecasting errors** of naive expectations, more agents will buy the RE forecast.

2-D dynamic system

- Pricing equation :

$$p_t = \frac{a - n_{2t} s p_{t-1}}{d + n_{1t} s} = \frac{2a - (1 - m_t) s p_{t-1}}{2d + (1 + m_t) s}.$$

- Evolutionary selection

$$m_{t+1} = \text{Tanh}(\frac{\beta}{2}[\frac{s}{2}(p_t - p_{t-1})^2 - C]).$$

- **Note Timing**:
  1. Old fractions determine market prices.
  2. Realized market prices determine new fractions.

2-D dynamic system in deviations

deviation from RE fundamental price:

$$x_t = p_t - p^*$$

$$x_t = \frac{-(1-m_t)s x_{t-1}}{2d + (1+m_t)s}$$

$$m_{t+1} = \text{Tanh}\left(\frac{\beta}{2}\left[\frac{s}{2}(x_t - x_{t-1})^2 - C\right]\right).$$

Properties of the 2-D Dynamics

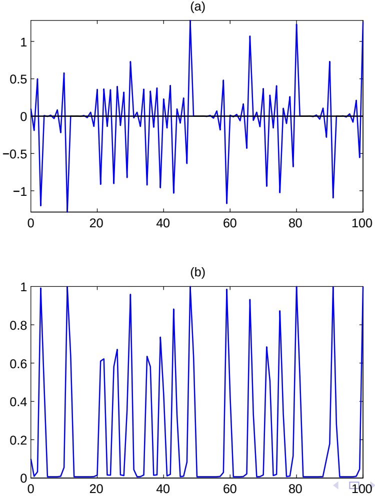
- If all agents are **rational**, then $p_t \equiv p^* = a/(d+s)$.
- If all agents are **naive**, then $p_t = \frac{a - sp_{t-1}}{d}$.
- Unique **steady state** $E = (p^*, m^*)$, $m^* = \mathrm{Tanh}(-\beta C/2)$.
- If $p_{t-1} = p^*$, then $p_t = p^*$ and $m_t = m^*$:
  stable manifold contains vertical line through steady state.
- If $s/d < 1$, then **globally stable** steady state

"Neo-classical" limit case: $\beta = \infty$

- If $s/d > 1$, $C > 0$ and $\beta = +\infty$ then
  - Steady state is **locally unstable saddle point**.
  - Steady state is **globally stable**.
- **Important note:** homoclinic orbits!!

Chaotic Dynamics

Irregular switching between cheap destabilizing free riding and costly sophisticated stabilizing predictor

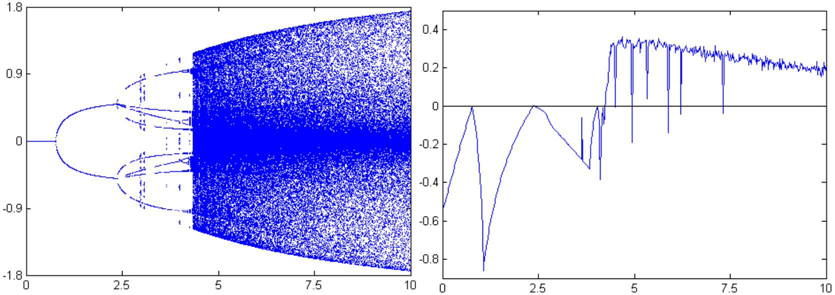


Rational Route to Randomness

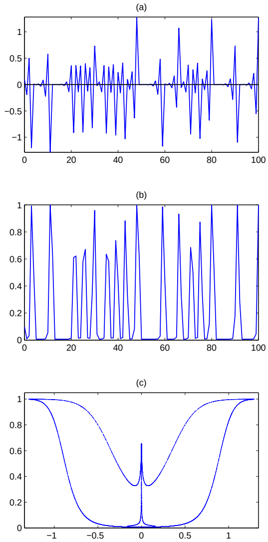
If $s/d > 1$ and $C > 0$, then

- $0 \leq \beta < \beta_1^*$: stable steady state
- $\beta = \beta_1^*$: **period doubling bifurcation**
- $\beta_1^* \leq \beta < \beta_2^*$: stable 2-cycle
- $\beta = \beta_2^*$: secondary period doubling bifurcations
- $\beta_2^* < \beta^* < \beta_3^*$: two **co-existing** stable 4-cycles
- $\beta > \beta_3^*$: complicated **chaotic dynamics**, strange attractors

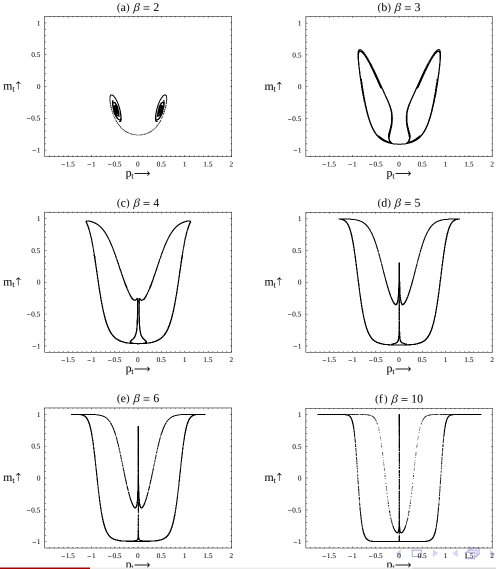
Rational versus naive: Rational Route to Randomness



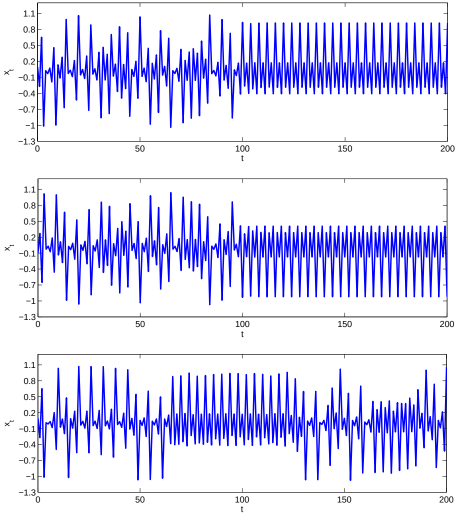
Rational versus naive: time series + attractors



Rational versus naive: unstable manifolds

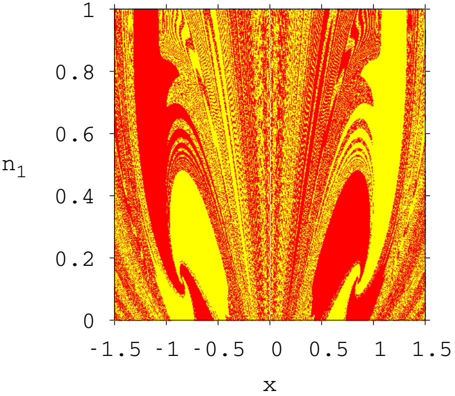


Rational versus naive: two co-existing stable 4-cycles



Basins of Attraction of two coexisting stable 4-cycles

fractal basin boundaries



Discrete choice model, with **asynchronous** updating

$$n_{ht} = (1-\delta)\frac{e^{\beta U_{h,t-1}}}{Z_{t-1}} + \delta n_{h,t-1},$$

where $Z_{t-1} = \sum e^{\beta U_{h,t-1}}$ is normalization factor,
$U_{h,t-1}$ **past strategy performance**, e.g. (weighted average) **past profits**

δ is **probability of not updating**
β is the **intensity of choice**.
$\beta = 0$: **all types equal weight** (in long run)
$\beta = \infty$: fraction $1-\delta$ **switches to best predictor**

Cobweb Model with Heterogeneous Beliefs

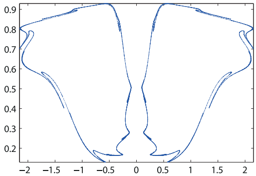
market clearing $a - dp_t = n_{1t} s p^e_{1t} + n_{2t} s p^e_{2t} (+\epsilon_t)$

n_{1t} and $n_{2t} = 1 - n_{1t}$ fractions of two types
forecasting rules:
rational/**fundamentalists**/**contrarians**/**SAC-learning** at cost $C > 0$ versus free **naive**

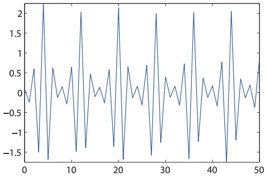
$$
\begin{aligned}
p^e_{1t} &= p_t && \text{rational} \\
&= p^* && \text{fundamentalist} \\
&= p^* + \beta(p_{t-1} - p^*) && \text{contrarian}, -1 < \beta < 0 \\
&= \alpha_{t-1} + \beta_{t-1}(p_{t-1} - \alpha_{t-1}) && \text{SAC-learning} \\
p^e_{2t} &= p_{t-1} && \text{naive}
\end{aligned}
$$

Fundamentalists versus naive

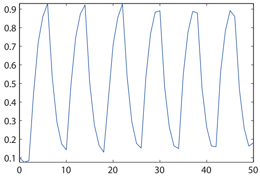
(x_t, n_{1t}) phase space

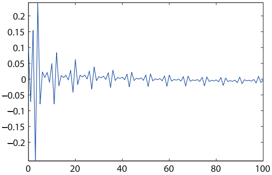


price deviations

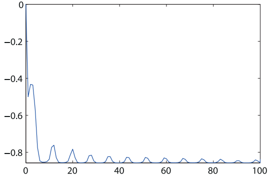


fraction fundamentalists





sample average



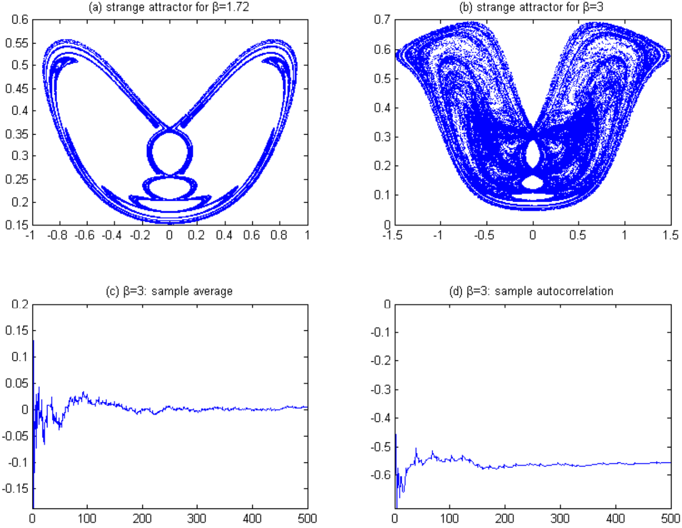
sample autocorrelation

Fundamentalists versus naive (continued)

- chaotic price fluctuations (when intensity of choice large)
- **sample average** of prices close to fundamental price
- strong **negative** first order autocorrelation in prices ($\beta_t \to -0.85$)

Question: will boundedly rational agents detect negative AC?
Replace fundamentalists by contrarians

Contrarians versus naive

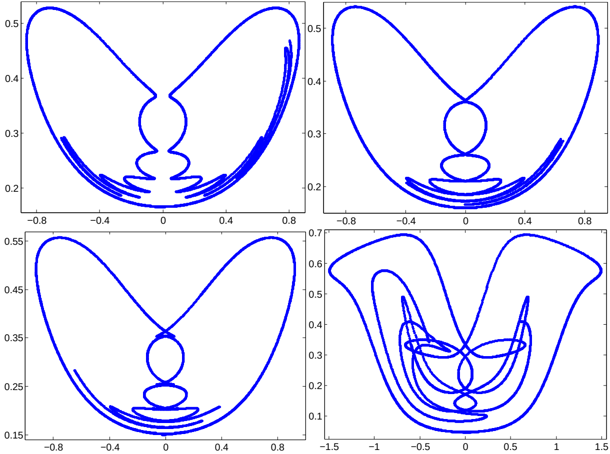


Contrarians versus naive (continued)

- chaotic price fluctuations (when intensity of choice large)
- **sample average** of prices close to fundamental price
- less strong **negative** first order autocorrelation in prices ($\beta_t \to -0.57$, with $\beta = -0.85$)

Question: can boundedly rational agents learn the correct negative AC?
Replace contrarians by SAC-learning

Contrarians versus naive: homoclinic intersections



Behavioral Sample Auto-Correlation (SAC) Learning

Hommes and Sorger, 1998

simple AR1 forecasting rule

$$p_t^e = \alpha_{t-1} + \beta_{t-1}(p_{t-1} - \alpha_{t-1})$$

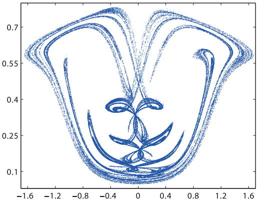
sample average after t periods:

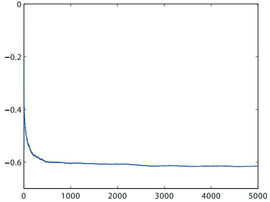
$$\alpha_{t-1} = \frac{1}{t}\sum_{i=0}^{t-1} p_i\,, \qquad t \geq 2$$

the **sample autocorrelation coefficient** at the first lag, after t periods:

$$\beta_{t-1} = \frac{\sum_{i=0}^{t-2}(p_i - \alpha_{t-1})(p_{i+1} - \alpha_{t-1})}{\sum_{i=0}^{t-1}(p_i - \alpha_{t-1})^2}\,, \qquad t \geq 2$$

SAC-learning versus naive





agents **learn to be contrarians**, with first order AC $\beta_t \to -0.62$

part of the (linear) structure has been "**arbitraged away**"

- fundamentalists: correct sample average
- contrarians: correct SAV + SAC

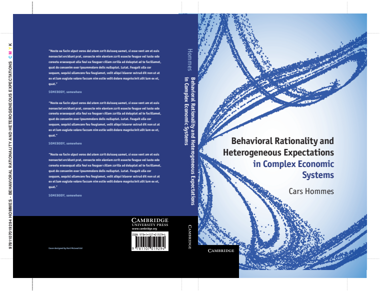
Note: adding more rules removes autocorrelations as in experiments

Summary Nonlinear Cobweb Model

- in **nonlinear** cobweb model with monotonic demand and supply, simple expectation rules may generate **chaos** in prices and errors;
- simple rules in a nonlinear world may be **behaviorally rational**
- **heterogeneous expectations** driven by recent performance may lead to homoclinic bifurcations and chaos
- **simple rules survive evolutionary competition**, especially when more sophisticated rules are costly

Questions?

Read the book
or ask them now!!



Thanks!