Complex Systems Workshop
Lecture I: Non-linear Dynamics, Chaos, Bifurcation & Strange Attractors

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Outline

Quadratic Map

Example of one-dimensional system:

$$x_{t+1} = f_{\lambda(x_t)} = \lambda x_t(1 - x_t) \tag{2.2}$$

1. initial state $x_0 \in [0, 1]$
2. **parameter** λ, $0 \leq \lambda \leq 4$.
3. **Problem**: what do the **orbits** look like?

Convergence to a steady state

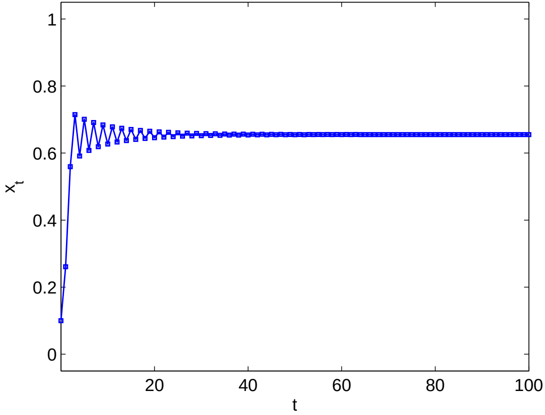


Figure: $\lambda = 2.9$ and $x_0 = 0.1$

Convergence to a 2-cycle

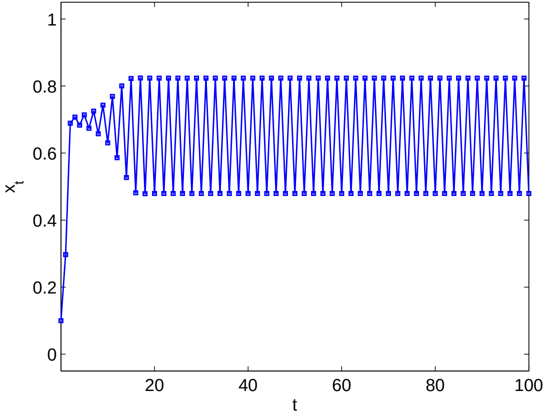


Figure: $\lambda = 3.3$ and $x_0 = 0.1$

Convergence to a 4-cycle

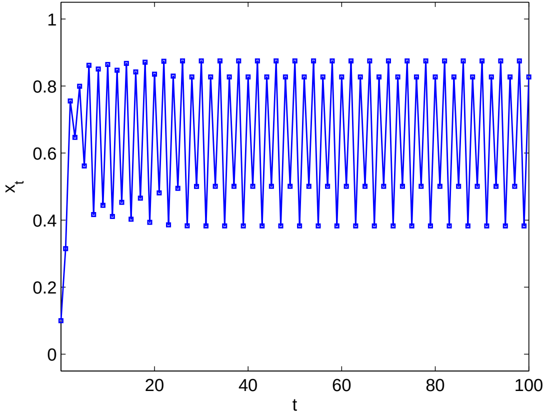


Figure: $\lambda = 3.5$ and $x_0 = 0.1$

Convergence to a 3-cycle

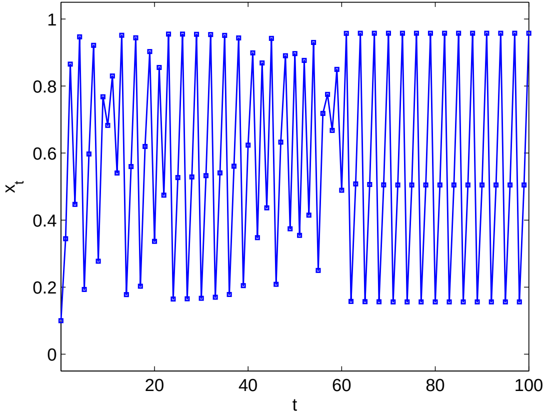


Figure: $\lambda = 3.83$ and $x_0 = 0.1$

Sensitive dependence on initial conditions

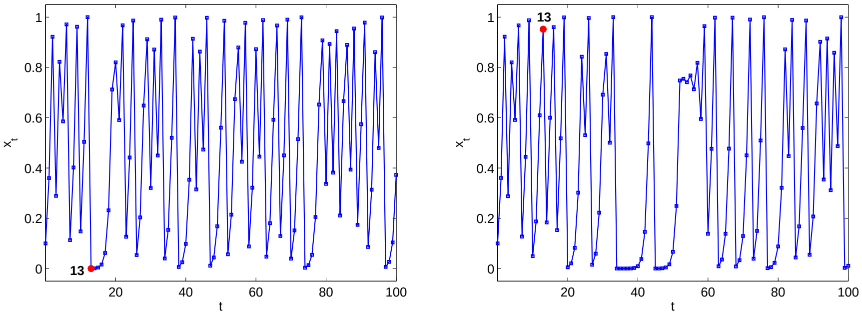


Figure: $\lambda = 4$ and $x_0 = 0.1$ (left) and (f) $\lambda = 4$ and $x_0 = 0.1001$ (right).

Periodic Orbits and Stability

A point x is called a **periodic point with period k** if

$$f^k(x) = x \quad \text{and} \quad f^i(x) \neq x, \quad 0 < i < k.$$

(Note: periodic point with period k is fixed point of k-th iterate f^k)

$\{x_1, x_2, ..., x_k\} = \{x_1, f(x_1), f^2(x_1), ...f^{k-1}(x_1)\}$ **periodic orbit** or **k-cycle**.

If x_i stable fixed point of f^k, then $\{x_1, x_2, ..., x_k\}$ **stable periodic orbit**; from the chain rule we have

$$\begin{aligned}(f^k)'(x_i) = (f^k)'(x_1) &= f'(f^{k-1}(x_1)) \cdot f'(f^{k-2}(x_1))....f'(f(x_1)) \cdot f'(x_1) \\ &= \prod_{i=0}^{k-1} f'(f^i(x_1)).\end{aligned}$$

(Note: $(f^k)'(x_j)$ is the product of derivatives along the orbit)

Aperiodic Point

A point x is called an **aperiodic point** if

1. the orbit of x is bounded,
2. the orbit of x is not periodic, and
3. the orbit of x does not converge to a periodic orbit

Bifurcation diagram of the quadratic map

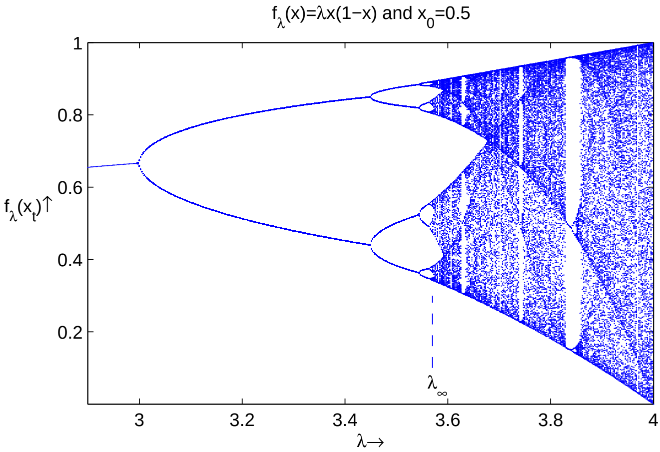


Figure: Bifurcation diagram of the quadratic map.

Period doubling bifurcation at $\lambda = 3$

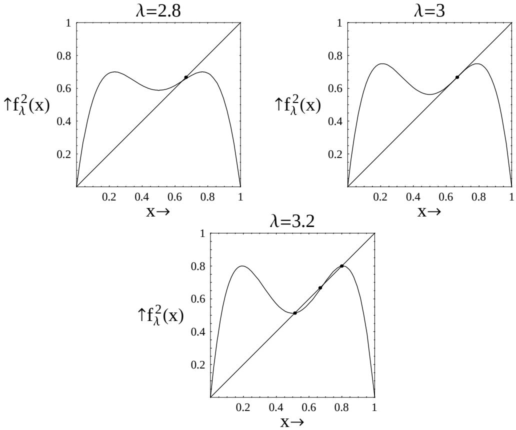


Figure: Graphs of the second iterate f^2 for three different λ-values close to the period-doubling bifurcation at $\lambda = 3$.

Tangent bifurcation for $x_{t+1} = x_t^2 + c$ at $c = 1/4$

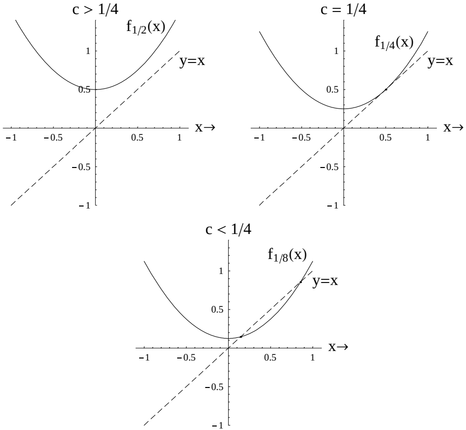


Figure: Tangent bifurcation for $x_{t+1} = x_t^2 + c$ at $c = 1/4$.

Creation of a 3-cycle by tangent bifurcation

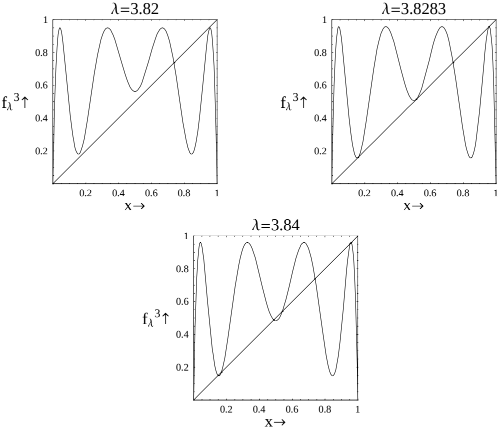


Figure: Creation of 3-cycle by tangent bifurcation at $\lambda \approx 3.8283$.

Tangent bifurcation of a 3-cycle in the quadratic map

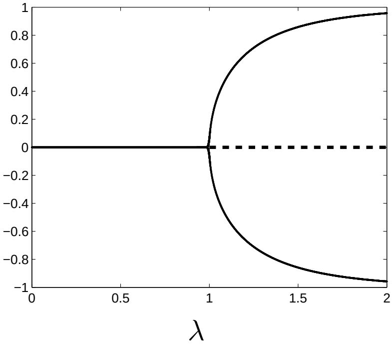
Proposition 2.1

For $\lambda = \lambda^* \approx 3.8283$ f_λ has a tangent bifurcation in which two 3-cycles are created, one stable and one unstable. Equivalently, at $\lambda = \lambda^*$ the third iterate f_λ^3 has a tangent bifurcation in which simultaneously 6 steady states are created, 3 stable and 3 unstable. We have
(1) for $\lambda < \lambda^* : f_\lambda$ has no 3-cycle,
(2) for $\lambda = \lambda^* : f_\lambda$ has one 3-cycle $\{x_1, x_2, x_3\}$, and $(f^3)'(x_i) = +1$, for $1 \leq i \leq 3$,
(3) for $\lambda > \lambda^*$ (and λ close to λ^*): f_λ has two 3-cycles, one stable and one unstable.

Pitchfork Bifurcation

Example: symmetric S-shaped map

$$x_{t+1} = \frac{e^{\lambda x_t} - e^{-\lambda x_t}}{e^{\lambda x_t} + e^{-\lambda x_t}}$$



Definition of topoligical chaos

The dynamics of a difference equation $x_{t+1} = f(x_t)$ is called **(topologically) chaotic** if the following three properties are satisfied:

1. There exists an infinite set P of (unstable) periodic points with different periods.
2. There exists an uncountable set S of aperiodic points (i.e. poinst whose orbits are bounded, not periodic and not converging to a periodic orbit).
3. f has sensitive dependence on initial conditions w.r.t. $\Lambda = P \cup S$, that is, there exists a positive distance C such that for all initial states $x_0 \in \Lambda$ and any ε-neighbourhood U of x_0, there exists an initial state $y_0 \in \Lambda \cap U$ and a time $T > 0$ such that the distance $d(x_T, y_T) = d(f^T(x_0), f^T(y_0)) > C$.

Example of Chaos: quadratic map for $\lambda = 4$

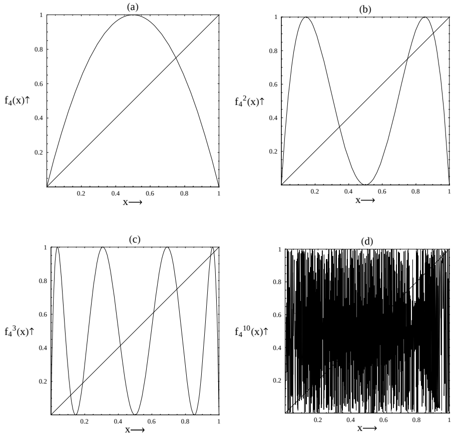


Figure: Graphs of (a) $f_4(x) = 4x(1-x)$, (b) f_4^2, (c) f_4^3 and (d) f_4^{10}.

Properties of quadratic map f_4

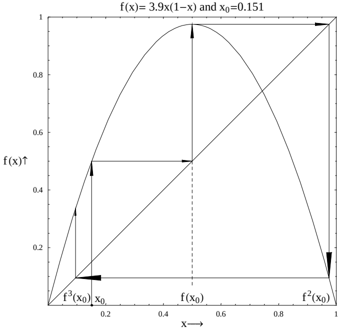
It can be shown that for any n, the graph of f_4^n has the following properties:

1. f_4^n has 2^{n-1} maxima equal to 1 and $2^{n-1}+1$ minima equal to 0 (including minima at $x=0$ and $x=1$).
2. f_4^n 'oscillates' 2^{n-1} times on the interval $[0,1]$.
3. the map f_4^n has 2^n fixed points.
4. for any interval I of arbitrarily small length ε, there exists an $N>0$ such that I contains points x, y with $f_4^N(x)=0$ and $f_4^N(y)=1$.

Period three implies chaos

Theorem 1

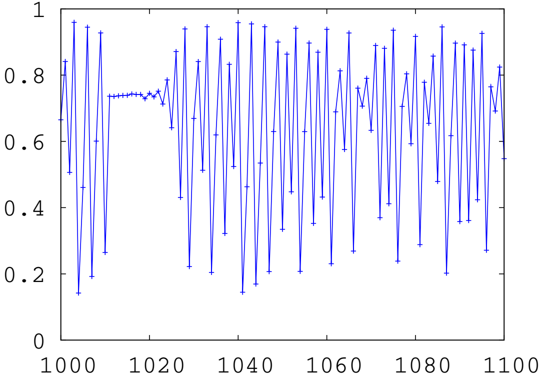
("Period 3 implies Chaos", Li & Yorke [1975]). Let $x_{t+1} = f(x_t)$ *be a 1-D difference equation with* f *a continuous map. If there exist a point* x_0 *such that* $f^3(x_0) \le x_0 < f(x_0) < f^2(x_0)$ *(or with* $>$ *instead of* $<$*) then the dynamics is topologically chaotic*



Topological Chaos with Noise

Quadratic map with small noise

$$x_{t+1} = 3.83x_t(1 - x_t)$$



Definition of true chaos

The dynamics of a difference equation $x_{t+1} = f(x_t)$ is called **'truly' chaotic** if there exists a set Λ of positive Lebesgue measure, such that f has sensitive dependence on initial conditions w.r.t. Λ, that is, there exists a positive distance C such that for all initial states $x_0 \in \Lambda$ and any ε-neighbourhood U of x_0, there exists an initial state $y_0 \in \Lambda \cap U$ and a time $T > 0$ such that the distance $d(x_T, y_T) = d(f^T(x_0), f^T(y_0)) > C$.

Lyapunov exponents

- The **Lyapunov exponent** is defined as $\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \ln(\mid f'(f^i(x_0)) \mid)$.
- Derivation:

$$\mid f^n(x_0+\delta) - f^n(x_0) \mid \approx \mid (f^n)'(x_0)\delta \mid = \; e^{n\lambda(x_0)} \mid \delta \mid$$

$$\Leftrightarrow e^{n\lambda(x_0)} = \mid (f^n)'(x_0) \mid \Rightarrow \lambda(x_0) = \frac{1}{n}\ln(\mid (f^n)'(x_0) \mid).$$

- The Lyapunov exponent measures the *average rate of divergence of nearby initial states*. It is the average of the logs of the absolute values of the derivative along the orbit.

Lyapunov exponent plot of the quadratic map

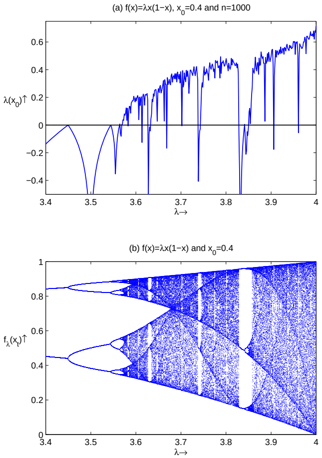


Figure: Lyapunov exponent L as a function of the parameter λ.

The asymmetric tent map

The asymmetric tent map T_β is the continuous, piecewise linear map $T_\beta : [0,1] \to [0,1]$ defined as

$$T_\beta(x) = \begin{cases} \frac{2}{1+\beta} x, & 0 \le x \le \frac{(\beta+1)}{2} \\ \\ \frac{2}{1-\beta}(1-x), & \frac{\beta+1}{2} < x \le 1, \end{cases} \tag{1}$$

where the parameter $-1 < \beta < +1$.

Graph of asymmetric tent maps

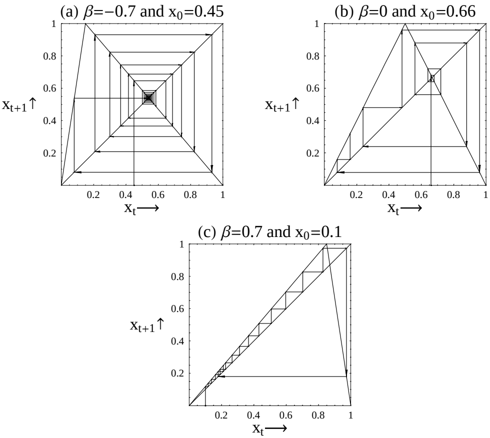


Figure: Graphs of the asymmetric tent map: (a) $\beta = -0.7$, (b) $\beta = 0$ and (c) $\beta = 0.7$

The properties of the asymmetric tent map

The piecewise linear difference equation $x_{t+1} = T_\beta(x_t)$ has the following properties:

1. For any integer $j \geq 1$, T_β has a periodic point of period j; all periodic orbits are unstable.
2. For Lebesgue almost all initial states $x_0 \in [0, 1]$, the time path $\{x_t\}_{t=0}^{\infty}$ is chaotic and dense in the interval $[0, 1]$.
3. For Lebesgue almost all initial states $x_0 \in [0, 1]$, the sample average of the (chaotic) time path is $\bar{x} = \lim_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} x_t = 1/2$.
4. For Lebesgue almost all initial states $x_0 \in [0, 1]$, the sample autocorrelation coefficient at lag j is $\rho_j = \beta^j$.

Two-dimensional (2-D) systems

$$(x_{t+1}, y_{t+1}) = F_\lambda(x_t, y_t),$$

F_λ nonlinear 2-D map and λ is a parameter.
The *orbit* with *initial state* (x_0, y_0) is the set

$$\{(x_0, y_0), (x_1, y_1), (x_2, y_2), ...\} = \{(x_0, y_0), F_\lambda(x_0, y_0), F_\lambda^2(x_0, y_0), ...\}.$$

Problem: what do these orbits look like and how does it depend on initial states and parameters?
Example: Hénon map:

$$\begin{aligned} x_{t+1} &= 1 - ax_t^2 + y_t \\ y_{t+1} &= bx_t, \end{aligned}$$

where a and b are parameters.
(special case $b = 0$ yields 1-D quadratic map)

Strange attractor of the Hénon map

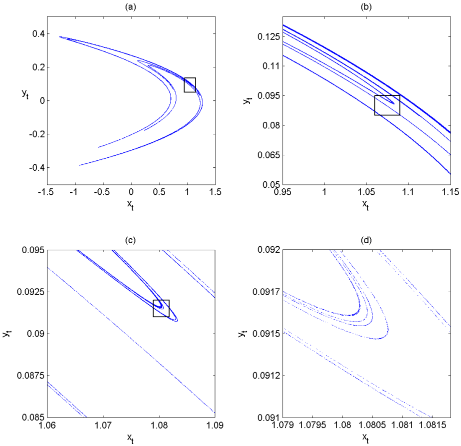


Figure: The strange attractor for the Hénon map $H_{a,b}$ with $a = 1.4$ and $b = 0.3$.

Chaotic time-series and SDIC

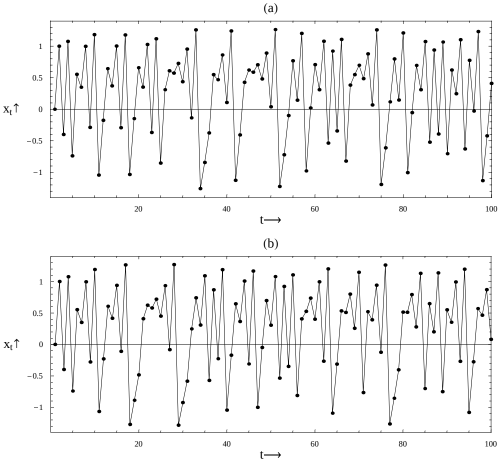


Figure: Chaotic time series and sensitive dependence for the Hénon map $H_{a,b}$ with $a = 1.4$ and $b = 0.3$. (a) $(x_0, y_0) = (0, 0)$ and (b) $(x_0, y_0) = (0.001, 0)$.

Attractor and Strange Attractor

An **attractor** of a K-dimensional system $X_{t+1} = F(X_t)$ is a compact set A with the following properties:

1. The set A is invariant, i.e. $F(A) \subset A$.
2. There exists an open neighborhood U of A (i.e. $A \subset U$), such that all initial states X_0 converge to the attractor A, i.e. for all $X_0 \in U$, $\lim_{n\to\infty} dist(F^n(X_0), A) = 0$.
3. There exists an initial state $X_0 \in A$ for which the orbit is dense in A.

An attractor A is called a **strange attractor** of the N-dimensional dynamical system $x_{t+1} = F(x_t)$, if the map F has sensitive dependence w.r.t. the set of initial states converging to A.

The delayed logistic map

- Delayed logistic map: $N_{t+1} = aN_t(1 - N_{t-1})$.
- Equivalently ($x_t = N_t$ and $y_t = N_{t-1}$):

$$\begin{aligned} x_{t+1} &= y_t \\ y_{t+1} &= ay_t(1 - x_t). \end{aligned}$$

- steady states

$$(x_1, y_1) = (0, 0) \qquad \text{and} \qquad (x_2, y_2) = (\frac{a-1}{a}, \frac{a-1}{a}).$$

- The eigenvalues of the system are $\lambda_1 = \frac{1}{2} - \frac{1}{2}\sqrt{5-4a}$ and $\lambda_2 = \frac{1}{2} + \frac{1}{2}\sqrt{5-4a}$.

Dynamical properties of the delayed logistic map

The eigenvalues λ_1 and λ_2 of $JF_a(\frac{a-1}{a}, \frac{a-1}{a})$ satisfy the following properties:

- $0 \leq a < 1$: real eigenvalues with $-1 < \lambda_1 < 1 < \lambda_2$, so $(\frac{a-1}{a}, \frac{a-1}{a})$ is a **saddle**.
- $1 < a < \frac{5}{4}$: real eigenvalues with $0 < \lambda_1 < \lambda_2 < 1$, so $(\frac{a-1}{a}, \frac{a-1}{a})$ is attracting (**stable node**).
- $\frac{5}{4} < a < 2$: complex eigenvalues with $\lambda_1\lambda_2 = a - 1 < 1$, so $(\frac{a-1}{a}, \frac{a-1}{a})$ is a **stable focus**.
- $a > 2$: complex eigenvalues with $\lambda_1\lambda_2 = a - 1 > 1$, so $(\frac{a-1}{a}, \frac{a-1}{a})$ is an **unstable focus**.
- **Hopf bifurcation** (or Neimark-Sacker) for $a = 2$
  complex eigenvalues on the unit circle

Attractors delayed logistic map

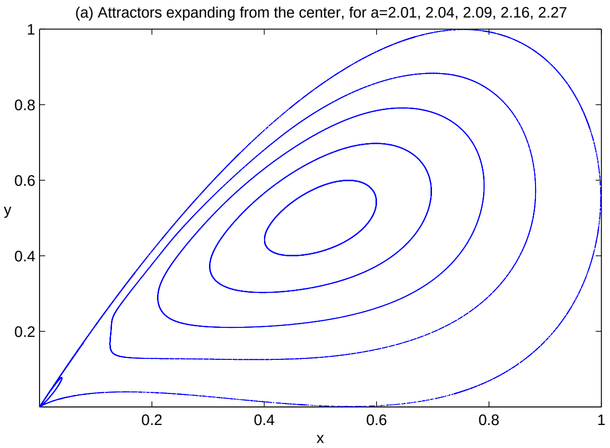


Figure: (a) Attractors for the logistic delayed equation for different a-values after the Hopf bifurcation.

Strange attractor of the delayed logisitic map

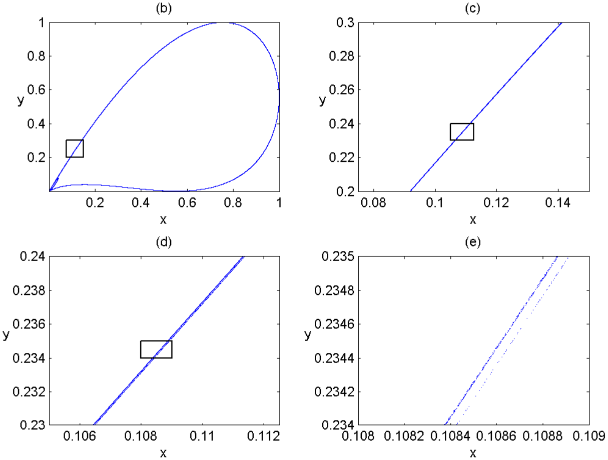


Figure: (b-e) The strange attractor for $a = 2.27$ and some enlargements.

Time series for the logistic delayed equation

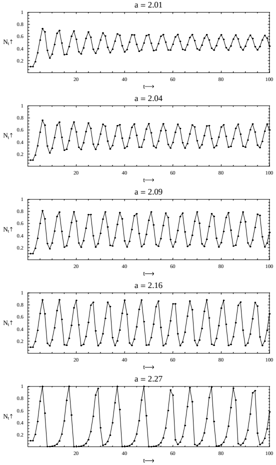


Figure: Time series for for different values of the parameter a.

Bifurcation diagrams for logistic delayed equation

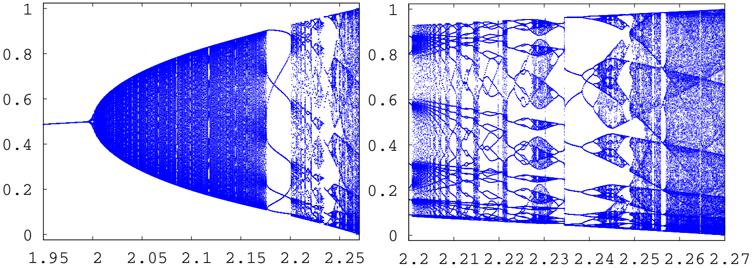


Figure: Hopf-bifurcation and breaking of an invariant circle bifurcation route to chaos

Types of Local Bifurcations

1. $\lambda = +1$:
   - *saddle-node bifurcation*: two new steady states, a saddle and a node;
   - *pitchfork bifurcation*: one steady state becomes unstable and two new stable steady states;
   - *transcritical bifurcation*; two steady states collide and exchange stability;
2. $\lambda = -1$:
   - *period-doubling bifurcation*: steady state loses stability and new stable 2-cycle;
3. a pair of complex eigenvalues λ_1 and λ_2 on the unit circle, i.e. $|\lambda_1 \lambda_2| = 1$:
   - *Hopf bifurcation*: steady state becomes an unstable focus and an attracting invariant circle emerges with (quasi-)periodic dynamics

Local (un)stable manifolds

Let p be a fixed point of the 2-D map F.
The *local stable manifold* and *local unstable manifold* of p are defined as

$$W^s_{loc}(p) = \{x \in U | \lim_{n \to \infty} F^n(x) = p\} \tag{2}$$

$$W^u_{loc}(p) = \{x \in U | \lim_{n \to -\infty} F^n(x) = p\} \tag{3}$$

where U is some small neighbourhood of p.

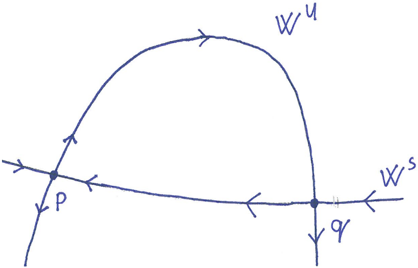
Global(un)stable manifolds

The *global stable manifold* and the *global unstable manifold* are now defined as

$$W^s(p) = \bigcup_{n=0}^{\infty} F^{-n}(W^s_{loc}) \quad (4)$$

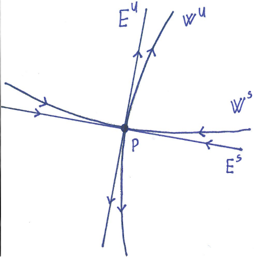
$$W^u(p) = \bigcup_{n=0}^{\infty} F^{n}(W^u_{loc}). \quad (5)$$

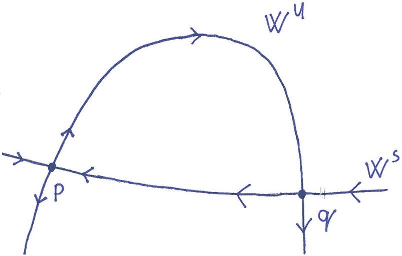
Homoclonic point



A point q is called a **homoclinic point** if $q \neq p$ and q is an intersection point of the stable and unstable manifolds of the saddle point p, that is, $q \in W^s(p) \bigcap W^u(p)$.

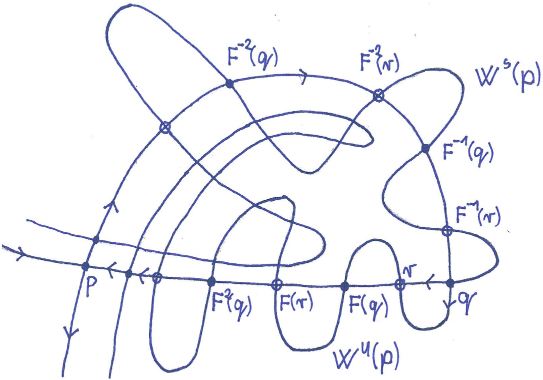
Henry Poincaré, ca. 1890: motion in the **three body problem** is unpredictable and chaotic





2-D map of suitable plane section has **homoclinic orbit**

Henry Poincaré, ca. 1890: homoclinic orbit implies sensitive dependence on initial conditions

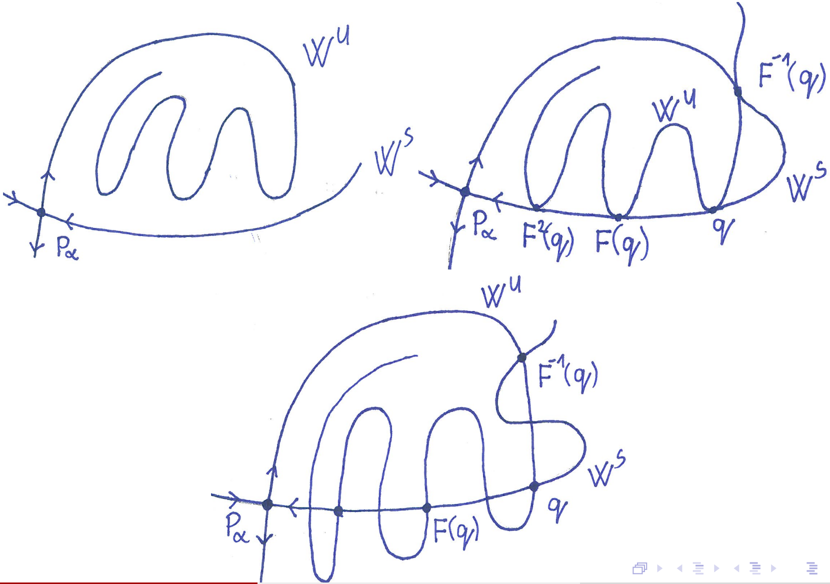


Homoclonic bifurcation

We say that F_α has a **homoclinic bifurcation**, associated to the saddle point p_α, at $\alpha = \alpha_0$, if

1. for $\alpha < \alpha_0$, $W^s(p_\alpha)$ and $W^u(p_\alpha)$ have no intersection point $q \neq p$;
2. for $\alpha = \alpha_0$, $W^s(p_\alpha)$ and $W^u(p_\alpha)$ have a point of homoclinic tangency;
3. for $\alpha > \alpha_0$, $W^s(p_\alpha)$ and $W^u(p_\alpha)$ have a transversal homoclinic intersection point.

Homoclonic bifurcation



Lyapunov exponents

- Consider a dynamic model $x_{t+1} = F(x_t)$, where F is an n-dimensional map.
- Let x_0 be an intial state vector and δ an initial perturbation vector.
- After n time periods, the separation between the two initial state vectors x_0 and $x_0 + \delta$ is approximately

$$\| F^n(x_0 + \delta) - F^n(x_0) \| \approx \| (D_{x_0} F^n)(\delta) \| .$$

- The Lyapunov exponent is a measure of the average exponential rate of divergence:

$$\lambda(x_0, \delta) = \lim_{n \to \infty} \frac{1}{n} ln(\| (D_{x_0} F^n)(\delta) \|).$$

Lyapunov exponents λ_1 and λ_2

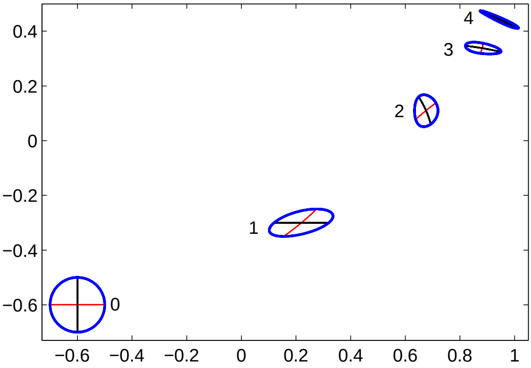
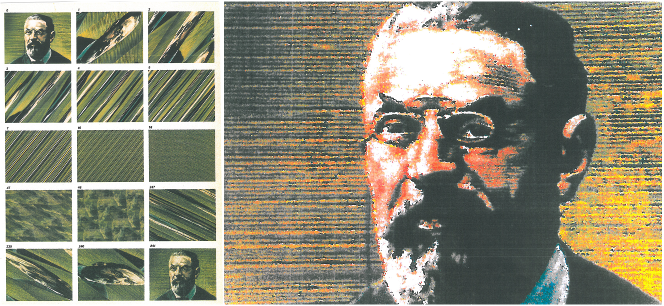


Figure: Lyapunov exponents λ_1 and λ_2

Henry Poincaré to Léon Walras (30 September, 1901):

"*You regard men as* **infinitely selfish** *and* **infinitely faresighted**. *The first hypothesis can perhaps be admitted as a first approximation, but the second should perhaps be regarded with some reservations*"



nonlinear complex systems: limits to predictability and rationality

Léon Walras to Henry Poincaré (3 October, 1901):

In reality, agents are neither infinitely selfish nor infinitely clairvoyant. Theory should indicate these frictions carefully ...

Implications Nonlinear Dynamics for Economics

- The fact that (simple) nonlinear systems exhibit **complex dynamics** calls for **reservations about rational behavior**, in particular **rational expectations**;
- In a **nonlinear** world, simple **heuristics** that work reasonably well may be the best **boundedly rational** agents can achieve